International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN 2319-3972
Vol. 2, Issue 2, May 2013, 43-54

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

LINEAR VOLTERRA FUZZY INTEGRAL EQUATIONS SOLVED BY MODIFIED TRAPEZOIDAL METHOD

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ABSTRACT

In this paper, the numerical solutions of linear Volterra fuzzy integral equations of the second kind (VFIEs-2) have been investigated using modified trapezoidal method. First, this equation was transformed into a system of crisp one and then applying the modified trapezoidal method on the resulting system to transform it into an algebraic system which is solved to obtain the solution. Two numerical examples are given to show the efficiency of the method.

KEYWORDS: Volterra Fuzzy Integral Equations, Fuzzy Numbers, Fuzzy Functions, Modified Trapezoidal Method

INTRODUCTION

One of the methods for solving definite integrals is modified trapezoid method, which is obtained by using Hermitian interpolation [1]. Topics of fuzzy integral equations (FIEs) which growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recent years. The concept of integration of fuzzy functions was first introduced by Dubois and Prade [2] and investigated by Goestschel and Voxman [3], Kaleva [4], Nanda [5] and others. One of the first applications of fuzzy integration was given by Wu and Ma [6] who investigated the Fredholm fuzzy integral equations of the second kind (FFIEs-2). In 1972, Chang and Zadeh [7] first introduced the concept of fuzzy derivative, followed up ten years later by Dubois and Prade [8], who used the extension principle in their approach.

PRELIMINARIES

Definition 2.1 [4]: A fuzzy number is a fuzzy set $v{:}R^1 \to I = [0,1]$ which satisfies:

- v is upper semi continuous,
- $v(x) = 0$ outside some interval $[c, d]$,
- There are real numbers $a, b : c \le a \le b \le d$ for which
  - $v(x)$ is monotonic increasing on $[c, a]$;
  - $v(x)$ is monotonic decreasing on $[b, d]$;
  - $v(x) = 1,\ a \le x \le b$.

The set of all such fuzzy numbers is denoted by E^1.

Definition 2.2 [4]

Let V be a fuzzy set on R. V is called a fuzzy interval if:

- V is normal: there exists $x_0 \in R$ such that $V(x_0) = 1$;
- V is convex: for all $x, t \in R, 0 \leq \lambda \leq 1$, it holds that $V(\lambda x + (1-\lambda)t) \geq min\{V(x), V(t)\}$;
- V is upper semi-continuous: for any $x_0 \in R$, it holds that $V(x_0) \geq lim_{x \to x_0^{\pm}} V(x)$;
- $[V]^0 = CL\{x \in R : V(x) > 0\}$ is a compact subset of R.

The $\alpha - cut$ of a fuzzy interval V, with $0 < \alpha \leq 1$ is the crisp set

$$[V]^\alpha = \{x \in R : V(x) > \alpha\} \quad (1)$$

For a fuzzy interval V, its $\alpha - cuts$ are closed intervals in R. Let denote them by

$$[V]^\alpha = [\underline{V}(\alpha), \overline{V}(\alpha)] \quad (2)$$

An alternative definition or parametric form of a fuzzy number which yields the same E^1 is given by Kaleva [4] as follows:

Definition 2.3: An arbitrary fuzzy number v in the parametric form is represented by an ordered pair of functions $(\underline{v}(\alpha), \overline{v}(\alpha))$ which satisfy the following requirements:

- $\underline{v}(\alpha)$ is a bounded left-continuous non-decreasing function over $[0,1]$;
- $\overline{v}(\alpha)$ is a bounded left-continuous non-increasing function over $[0,1]$;
- $\underline{v}(\alpha) \leq \overline{v}(\alpha),\ 0 \leq \alpha \leq 1$.

For arbitrary fuzzy numbers $v = (\underline{v}(\alpha), \overline{v}(\alpha))$, $w = (\underline{w}(\alpha), \overline{w}(\alpha))$ and real number k, one may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as follows:

- (a) $v = w$ if and only if $\underline{v}(\alpha) = \underline{w}(\alpha)$ and $\overline{v}(\alpha) = \overline{w}(\alpha)$;
- (b) $v + w = (\underline{v}(\alpha) + \underline{w}(\alpha), \overline{v}(\alpha) + \overline{w}(\alpha))$;
- (c) $kv = \begin{cases} (k\underline{v}(\alpha), k\overline{v}(\alpha)), & k \geq 0 \\ (k\overline{v}(\alpha), k\underline{v}(\alpha)), & k < 0 \end{cases}$

Definition 2.4 [3]: For arbitrary fuzzy numbers $v = (\underline{v}(\alpha), \overline{v}(\alpha))$ and $w = (\underline{w}(\alpha), \overline{w}(\alpha))$ the quantity

$$D(v, w) = sup_{0 \leq \alpha \leq 1}\{max[|\underline{v}(\alpha) - \underline{w}(\alpha)|, |\overline{v}(\alpha) - \overline{w}(\alpha)|]\} \quad (3)$$

is the distance between v and w. It is shown [9] that (E^1, D) is a complete metric space.

Definition 2.5 [10]: A function $f: R^1 \to E^1$ is called a fuzzy function. A function f is said to be continuous if for arbitrary fixed $t_0 \in R^1$ and $\epsilon > 0$, a $\delta > 0$ exists such that

$$\text{if } |t - t_0| < \delta \text{ then } D[f(t), f(t_0)] < \epsilon \quad (4)$$

for each $t \in R^1$.

Definition 2.6 [11]: Let $f:(a,b) \to E^1$ be a fuzzy function and $t_0 \in (a,b)$. One can say that f is differentiable at t_0 if two forms were sustained as follows:

- It exists an element $f'(t_0) \in E^1$ such that, for all $h > 0$ sufficiently near to 0, there are $f(t_0+h) - f(t_0)$, $f(t_0) - f(t_0-h)$ and the limits:

$$\lim_{h\to 0^+} \frac{f(t_0+h)-f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0)-f(t_0-h)}{h} = \acute{f}(t_0) \quad (5)$$

- It exists an element $f'(t_0) \in E^1$ such that, for all $h < 0$ sufficiently near to 0, there are $f(t_0+h) - f(t_0)$, $f(t_0) - f(t_0-h)$ and the limits:

$$\lim_{h\to 0^-} \frac{f(t_0+h)-f(t_0)}{h} = \lim_{h\to 0^-} \frac{f(t_0)-f(t_0-h)}{h} = \acute{f}(t_0) \quad (6)$$

Theorem 2.1 [12]: Let $f:(a,b) \to E^1$ be a fuzzy function and denote

$[f(t)]^\alpha = [\underline{f}(x,\alpha), \overline{f}(x,\alpha)]$ for each $\alpha \in [0,1]$ and $x \in (0,1)$.

Then

- If f is differentiable in the first form 1, then $\underline{f}(x,\alpha)$ and $\overline{f}(x,\alpha)$ are differentiable functions and

$$[f'(t)]^\alpha = \left[\underline{f}'(x,\alpha), \overline{f}'(x,\alpha)\right] \quad (7)$$

- If f is differentiable in the second form 2, then $\underline{f}(x,\alpha)$ and $\overline{f}(x,\alpha)$ are differentiable functions and

$$[f'(t)]^\alpha = \left[\overline{f}'(x,\alpha), \underline{f}'(x,\alpha)\right] \quad (8)$$

Definition 2.7: Let $f:(a,b) \to E^1$ be a fuzzy function and $t_0 \in (a,b)$. f is strongly generalized differentiable at t_0, if there exists an element $f'(t_0) \in E^1$, such that

- for all $h > 0$ sufficiently small, there are $f(t_0+h) - f(t_0)$, $f(t_0) - f(t_0-h)$ and the limits (in the metric *D*)

$$lim_{h\to 0} \frac{f(t_0+h)-f(t_0)}{h} = lim_{h\to 0} \frac{f(t_0)-f(t_0-h)}{h} = \acute{f}(t_0) \quad (9)$$

Or;

- for all $h > 0$ sufficiently small, there are $f(t_0) - f(t_0+h)$, $f(t_0-h) - f(t_0)$ and the limits

$$lim_{h\to 0} \frac{f(t_0)-f(t_0+h)}{-h} = lim_{h\to 0} \frac{f(t_0-h)-f(t_0)}{(-h)} = \acute{f}(t_0) \quad (10)$$

- for all $h > 0$ sufficiently small, there are $f(t_0+h) - f(t_0)$, $f(t_0) - f(t_0-h)$ and the limits

$$lim_{h\to 0} \frac{f(t_0+h)-f(t_0)}{h} = lim_{h\to 0} \frac{f(t_0-h)-f(t_0)}{-h} = \acute{f}(t_0) \quad (11)$$

Or;

- for all $h > 0$ sufficiently small, there are $f(t_0) - f(t_0 + h)$, $f(t_0) - f(t_0 - h)$ and the limits

$$\lim_{h\to 0}\frac{f(t_0)-f(t_0+h)}{-h} = \lim_{h\to 0}\frac{f(t_0)-f(t_0-h)}{-h} = \acute{f}(t_0) \tag{12}$$

(h and $(-h)$ at denominators means $\frac{1}{h}\otimes$ and $-\frac{1}{h}\otimes$, respectively).

In this research , the following notations will be used:

$\underline{f}_i(\alpha) = \underline{f}(x_i,\alpha)$, $\overline{f}_i(\alpha) = \overline{f}(x_i,\alpha)$ $\underline{f}_i'(\alpha) = \underline{f}'(x_i,\alpha)$ $\overline{f}_i'(\alpha) = \overline{f}'(x_i,\alpha)$, $K_{ij} = K(x_i,x_j)$, $L_{ij} = L(x_i,x_j)$, $J_{ij} = J(x_i,x_j)$, $\underline{u}_i(\alpha) = \underline{u}(x_i,\alpha)$, $\overline{u}_i(\alpha) = \overline{u}(x_i,\alpha)$ $\underline{u}_i'(\alpha) = \underline{u}'(x_i,\alpha)$, $\overline{u}'_i(\alpha) = \overline{u}'(x_i,\alpha)$

where ; $L(x,t) = \frac{\partial}{\partial t}K(x,t)$ and $J(x,t) = \frac{\partial}{\partial x}K(x,t)$

NUMERICAL SOLUTIONS OF LINEAR VOLTERRA FUZZY INTEGRAL EQUATIONS OF THE SECOND KIND

Consider the linear Volterra fuzzy integral equation of the second kind

$$u(x) = f(x) + \int_a^x K(x,t)\,u(t)\,dt \tag{13}$$

where $f(x)$ is a given continuous fuzzy function on $[a,b]$, $K(x,t)$ is a given continuous kernel over the square $\Delta = \{(x,t): a \le t \le x \le b\}$ and $u(x)$ is unknown fuzzy function to be determined, the kernel $K(x,t) = 0$ for $t > x$. Substitution $x_i, 0 \le i \le n$ in equation (13) gives

$$u(x_i) = f(x_i) + \int_a^{x_i} K(x_i,t)\,u(t)\,dt \tag{14}$$

To solve equation (13) an approximation of the integral in the right hand side of equation (14) was carried out using modified trapezoidal method to obtain;

$$u(x_0) = u(a) = f(a) \tag{15}$$

$$u(x_i) = f(x_i) + \frac{h}{2}K_{00}u(x_0) + h\sum_{j=1}^{i-1}K_{ij}u(x_j) + \frac{h}{2}K_{ii}u(x_i) + \frac{h^2}{12}[L_{i0}u(x_0) + K_{i0}u'(x_0) - L_{ii}u(x_i) + K_{ii}u'(x_i)] \tag{16}$$

for $1 \le i \le n$, $h = \frac{b-a}{n}$. By using $\alpha - cuts$ form of fuzzy functions considering three cases, in all cases supposing that $K_{ij} \ge 0$ for $1 \le j \le k$, $K_{ij} \le 0$ for $k+1 \le j \le i \le n$ and $L_{ij} \ge 0$ for $0 \le j \le i \le n$, $J_{ij} \ge 0$ for $1 \le j \le k$ and $J_{ij} \le 0$ for $k+1 \le j \le i \le n$.

Case 1

If u_0 and u_i are differentiable in the first form 1 leading to;

$$\left[\underline{u}_0(\alpha), \overline{u}_0(\alpha)\right] = \left[\underline{f}_0(\alpha), \overline{f}_0(\alpha)\right]$$

$$\left[\underline{u}_i(\alpha),\overline{u}_i(\alpha)\right] = \left[\underline{f_i}(\alpha),\overline{f}_i(\alpha)\right] + \tfrac{h}{2}K_{i0}\left[\underline{u}_0(\alpha),\overline{u}_0(\alpha)\right] + h\sum_{j=1}^{k} K_{ij}\left[\underline{u}_j(\alpha),\overline{u}_j(\alpha)\right] +$$
$$h\textstyle\sum_{j=k+1}^{i-1} K_{ij}\left[\overline{u}_j(\alpha),\underline{u}_j(\alpha)\right] + \frac{h}{2}K_{ii}\left[\overline{u}_i(\alpha),\underline{u}_i(\alpha)\right] + \frac{h^2}{12}\Big[L_{i0}\left[\underline{u}_0(\alpha),\overline{u}_0(\alpha)\right] + K_{i0}\left[\underline{u}_0'(\alpha),\overline{u}_0'(\alpha)\right] -$$
$$L_{ii}\left[\overline{u}_i(\alpha),\underline{u}_i(\alpha)\right] + K_{ii}\left[\overline{u}_i'(\alpha),\underline{u}_i'(\alpha)\right]\Big] \quad (17)$$

Then obtaining the;

$$\underline{u}_0(\alpha) = \underline{f_0}(\alpha)$$
$$\underline{u}_i(\alpha) = \underline{f_i}(\alpha) + \tfrac{h}{2}K_{i0}\,\underline{u}_0(\alpha) + h\textstyle\sum_{j=1}^{k} K_{ij}\underline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} K_{ij}\overline{u}_j(\alpha) + \frac{h}{2}K_{ii}\overline{u}_i(\alpha) + \frac{h^2}{12}\big[L_{i0}\,\underline{u}_0(\alpha) +$$
$$K_{i0}\,\underline{u}_0'(\alpha) - L_{ii}\,\overline{u}_i(\alpha) + K_{ii}\overline{u}_i'(\alpha)\big] \quad (18)$$

$$\overline{u}_0(\alpha) = \overline{f}_0(\alpha)$$
$$\overline{u}_i(\alpha) = \overline{f}_i(\alpha) + \tfrac{h}{2}K_{i0}\,\overline{u}_0(\alpha) + h\textstyle\sum_{j=1}^{k} K_{ij}\overline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} K_{ij}\,\underline{u}_j(\alpha) + \frac{h}{2}K_{ii}\,\underline{u}_i(\alpha) + \frac{h^2}{12}\big[L_{i0}\,\overline{u}_0(\alpha) +$$
$$K_{i0}\,\overline{u}_0'(\alpha) - L_{ii}\,\underline{u}_i(\alpha) + K_{ii}\,\underline{u}_i'(\alpha)\big] \quad (19)$$

Case 2

If u_0 and u_i are differentiable in the second form 2, leading to;

$$\left[\underline{u}_0(\alpha),\overline{u}_0(\alpha)\right] = \left[\underline{f_0}(\alpha),\overline{f}_0(\alpha)\right]$$

$$\left[\underline{u}_i(\alpha),\overline{u}_i(\alpha)\right] = \left[\underline{f_i}(\alpha),\overline{f}_i(\alpha)\right] + \tfrac{h}{2}K_{i0}\left[\underline{u}_0(\alpha),\overline{u}_0(\alpha)\right] + h\sum_{j=1}^{k} K_{ij}\left[\underline{u}_j(\alpha),\overline{u}_j(\alpha)\right]$$

$$+h\textstyle\sum_{j=k+1}^{i-1} K_{ij}\left[\overline{u}_j(\alpha),\underline{u}_j(\alpha)\right] + \frac{h}{2}K_{ii}\left[\overline{u}_i(\alpha),\underline{u}_i(\alpha)\right] + \frac{h^2}{12}\Big[L_{i0}\left[\underline{u}_0(\alpha),\overline{u}_0(\alpha)\right] + K_{i0}\left[\overline{u}_0'(\alpha),\underline{u}_0'(\alpha)\right] -$$
$$L_{ii}\left[\overline{u}_i(\alpha),\underline{u}_i(\alpha)\right] + K_{ii}\left[\underline{u}_i'(\alpha),\overline{u}_i'(\alpha)\right]\Big] \quad (20)$$

Then obtaining the;

$$\underline{u}_0(\alpha) = \underline{f_0}(\alpha)$$

$$\underline{u}_i(\alpha) = \underline{f_i}(\alpha) + \tfrac{h}{2}K_{i0}\,\underline{u}_0(\alpha) + h\textstyle\sum_{j=1}^{k} K_{ij}\underline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} K_{ij}\overline{u}_j(\alpha) + \frac{h}{2}K_{ii}\overline{u}_i(\alpha) + \frac{h^2}{12}\big[L_{i0}\underline{u}_0(\alpha) +$$
$$K_{i0}\overline{u}_0'(\alpha) - L_{ii}\overline{u}_i(\alpha) + K_{ii}\underline{u}_i'(\alpha)\big] \quad (21)$$

$$\overline{u}_0(\alpha) = \overline{f}_0(\alpha)$$
$$\overline{u}_i(\alpha) = \overline{f}_i(\alpha) + \tfrac{h}{2}K_{i0}\overline{u}_0(\alpha) + h\textstyle\sum_{j=1}^{k} K_{ij}\overline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} K_{ij}\,\underline{u}_j(\alpha) + \frac{h}{2}K_{ii}\underline{u}_i(\alpha) + \frac{h^2}{12}\big[L_{i0}\overline{u}_0(\alpha) +$$
$$K_{i0}\underline{u}_0'(\alpha) - L_{ii}\underline{u}_i(\alpha) + K_{ii}\overline{u}_i'(\alpha)\big] \quad (22)$$

Case 3

If u_0 is differentiable in the first form 1 and u_i is differentiable in the second form 2 leading to;

$$\left[\underline{u}_0(\alpha),\overline{u}_0(\alpha)\right] = \left[\underline{f_0}(\alpha),\overline{f}_0(\alpha)\right]$$

$$\left[\underline{u}_i(\alpha),\overline{u}_i(\alpha)\right]=\left[\underline{f}_i(\alpha),\overline{f}_i(\alpha)\right]+\frac{h}{2}K_{i0}\left[\underline{u}_0(\alpha),\overline{u}_0(\alpha)\right]+h\sum_{j=1}^{k}K_{ij}\left[\underline{u}_j(\alpha),\overline{u}_j(\alpha)\right]+h\sum_{j=k+1}^{i-1}K_{ij}\left[\overline{u}_j(\alpha),\underline{u}_j(\alpha)\right]$$

$$+\frac{h}{2}K_{ii}\left[\overline{u}_i(\alpha),\underline{u}_i(\alpha)\right]+\frac{h^2}{12}\left[L_{i0}\left[\underline{u}_0(\alpha),\overline{u}_0(\alpha)\right]+K_{i0}\left[\underline{u}_0'(\alpha),\overline{u}_0'(\alpha)\right]-L_{ii}\left[\overline{u}_i(\alpha),\underline{u}_i(\alpha)\right]+K_{ii}\left[\underline{u}_i'(\alpha),\overline{u}_i'(\alpha)\right]\right] \quad (23)$$

Then obtaining the;

$$\underline{u}_0(\alpha)=\underline{f}_0(\alpha)$$

$$\underline{u}_i(\alpha)=\underline{f}_i(\alpha)+\frac{h}{2}K_{i0}\underline{u}_0(\alpha)+h\sum_{j=1}^{k}K_{ij}\,\underline{u}_j(\alpha)+h\sum_{j=k+1}^{i-1}K_{ij}\,\overline{u}_j(\alpha)+\frac{h}{2}K_{ii}\overline{u}_i(\alpha)+\frac{h^2}{12}\left[L_{i0}\underline{u}_0(\alpha)+K_{i0}\underline{u}_0'(\alpha)-L_{ii}\overline{u}_i(\alpha)+K_{ii}\underline{u}_i'(\alpha)\right] \quad (24)$$

$$\overline{u}_0(\alpha)=\overline{f}_0(\alpha)$$

$$\overline{u}_i(\alpha)=\overline{f}_i(\alpha)+\frac{h}{2}K_{i0}\overline{u}_0(\alpha)+h\sum_{j=1}^{k}K_{ij}\,\overline{u}_j(\alpha)+h\sum_{j=k+1}^{i-1}K_{ij}\,\underline{u}_j(\alpha)+\frac{h}{2}K_{ii}\underline{u}_i(\alpha)+\frac{h^2}{12}\left[L_{i0}\overline{u}_0(\alpha)+K_{i0}\overline{u}_0'(\alpha)-L_{ii}\underline{u}_i(\alpha)+K_{ii}\overline{u}_i'(\alpha)\right] \quad (25)$$

Differentiating equation (13) with respect to x to get;

$$u'(x)=f'(x)+K(x,x)u(x)+\int_a^x J(x,t)u(t)dt \quad (26)$$

Substitute x_i, $0\le i\le n$ in equation (26) to obtain;

$$u'(x_0)=f'(x_0)+K_{00}u(x_0) \quad (27)$$

$$u'(x_i)=f'(x_i)+K_{ii}u(x_i)+\int_a^{x_i} J(x_i,t)u(t)dt \quad (28)$$

for $1\le i\le n$ was consider to solve equation (28) by using repeated trapezoidal method to get;

$$u'(x_i)=f'(x_i)+K_{ii}u(x_i)+\frac{h}{2}J_{i0}u(x_0)+h\sum_{j=1}^{i-1}J_{ij}\,u(x_j)+\frac{h}{2}J_{ii}u(x_i) \quad (29)$$

Now, by using $\alpha-cuts$ form of fuzzy functions .the following three cases:

Case 1

If u_i and f_i are differentiable in the first form 1,the following leads to;

$$\left[\underline{u}'_0(\alpha),\overline{u}'_0(\alpha)\right]=\left[\underline{f}'_0(\alpha),\overline{f}'_0(\alpha)\right]+K_{00}\left[\underline{u}_0(\alpha),\overline{u}_0(\alpha)\right] \quad (30)$$

$$\left[\underline{u}'_i(\alpha),\overline{u}'_i(\alpha)\right]=\left[\underline{f}_i'(\alpha),\overline{f}'_i(\alpha)\right]+K_{ii}\left[\overline{u}_i(\alpha),\underline{u}_i(\alpha)\right]+\frac{h}{2}J_{i0}\left[\underline{u}_0(\alpha),\overline{u}_0(\alpha)\right]+h\sum_{j=1}^{k}J_{ij}\left[\underline{u}_j(\alpha),\overline{u}_j(\alpha)\right]+h\sum_{j=k+1}^{i-1}J_{ij}\left[\overline{u}_j(\alpha),\underline{u}_j(\alpha)\right]+\frac{h}{2}J_{ii}\left[\overline{u}_i(\alpha),\underline{u}_i(\alpha)\right] \quad (31)$$

Then obtaining the;

$$\underline{u}'_0(\alpha)=\underline{f}'_0(\alpha)+K_{00}\underline{u}_0(\alpha)$$

$$\underline{u}'_i(\alpha) = \underline{f_i}'^{(\alpha)} + K_{ii}\overline{u}_i(\alpha) + \frac{h}{2}J_{i0}\underline{u}_0(\alpha) + h\sum_{j=1}^{k} J_{ij}\,\underline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} J_{ij}\,\overline{u}_j(\alpha) + \frac{h}{2}J_{ii}\overline{u}_i(\alpha) \quad (32)$$

$$\overline{u}'_0(\alpha) = \overline{f}'_0(\alpha) + K_{00}\overline{u}_0(\alpha)$$

$$\overline{u}'_i(\alpha) = \overline{f}'_i(\alpha) + K_{ii}\underline{u}_i(\alpha) + \frac{h}{2}J_{i0}\overline{u}_0(\alpha) + h\sum_{j=1}^{k} J_{ij}\,\overline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} J_{ij}\,\underline{u}_j(\alpha) + \frac{h}{2}J_{ii}\underline{u}_i(\alpha) \quad (33)$$

Case 2

If u_i and f_i are differentiable in the second form 2, the following leads to:

$$\left[\overline{u}'_0(\alpha), \underline{u}'_0(\alpha)\right] = \left[\overline{f}'_0(\alpha), \underline{f}'_0(\alpha)\right] + K_{00}\left[\underline{u}_0(\alpha), \overline{u}_0(\alpha)\right] \quad (34)$$

$$\left[\overline{u}'_i(\alpha), \underline{u}'_i(\alpha)\right] = \left[\overline{f}'_i(\alpha), \underline{f_i}'(\alpha)\right] + K_{ii}\left[\overline{u}_i(\alpha), \underline{u}_i(\alpha)\right] + \frac{h}{2}J_{i0}\left[\underline{u}_0(\alpha), \overline{u}_0(\alpha)\right] + h\sum_{j=1}^{k} J_{ij}\left[\underline{u}_j(\alpha), \overline{u}_j(\alpha)\right] + h\sum_{j=k+1}^{i-1} J_{ij}\left[\overline{u}_j(\alpha), \underline{u}_j(\alpha)\right] + \frac{h}{2}J_{ii}\left[\overline{u}_i(\alpha), \underline{u}_i(\alpha)\right] \quad (35)$$

Then obtaining the;

$$\overline{u}'_0(\alpha) = \overline{f}'_0 + K_{00}\underline{u}_0(\alpha)$$

$$\overline{u}'_i(\alpha) = \overline{f}'_i(\alpha) + K_{ii}\overline{u}_i(\alpha) + \frac{h}{2}J_{i0}\underline{u}_0(\alpha) + h\sum_{j=1}^{k} J_{ij}\,\underline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} J_{ij}\,\overline{u}_j(\alpha) + \frac{h}{2}J_{ii}\overline{u}_i(\alpha) \quad (36)$$

$$\underline{u}'_0(\alpha) = \underline{f}'_0(\alpha) + K_{00}\overline{u}_0(\alpha)$$

$$\underline{u}'_i(\alpha) = \underline{f_i}'^{(\alpha)} + K_{ii}\underline{u}_i(\alpha) + \frac{h}{2}J_{i0}\overline{u}_0(\alpha) + h\sum_{j=1}^{k} J_{ij}\,\overline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} J_{ij}\,\underline{u}_j(\alpha) + \frac{h}{2}J_{ii}\underline{u}_i(\alpha) \quad (37)$$

Case 3: If u is differentiable in the first form 1 and f is differentiable in the second form 2, this yields;

$$\left[\underline{u}'_0(\alpha), \overline{u}'_0(\alpha)\right] = \left[\overline{f}'_0(\alpha), \underline{f}'_0(\alpha)\right] + K_{00}\left[\underline{u}_0(\alpha), \overline{u}_0(\alpha)\right] \quad (38)$$

$$\left[\underline{u}'_i(\alpha), \overline{u}'_i(\alpha)\right] = \left[\overline{f}'_i(\alpha), \underline{f_i}'(\alpha)\right] + K_{ii}\left[\overline{u}_i(\alpha), \underline{u}_i(\alpha)\right] + \frac{h}{2}J_{i0}\left[\underline{u}_0(\alpha), \overline{u}_0(\alpha)\right] + h\sum_{j=1}^{k} J_{ij}\left[\underline{u}_j(\alpha), \overline{u}_j(\alpha)\right] + h\sum_{j=k+1}^{i-1} J_{ij}\left[\overline{u}_j(\alpha), \underline{u}_j(\alpha)\right] + \frac{h}{2}J_{ii}\left[\overline{u}_i(\alpha), \underline{u}_i(\alpha)\right] \quad (39)$$

Also obtaining the;

$$\underline{u}'_0(\alpha) = \overline{f}'_0(\alpha) + K_{00}\underline{u}_0(\alpha)$$

$$\underline{u}'_i(\alpha) = \overline{f}'_i(\alpha) + K_{ii}\overline{u}_i(\alpha) + \frac{h}{2}J_{i0}\underline{u}_0(\alpha) + h\sum_{j=1}^{k} J_{ij}\,\underline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} J_{ij}\,\overline{u}_j(\alpha) + \frac{h}{2}J_{ii}\overline{u}_i(\alpha) \quad (40)$$

$$\overline{u}'_0(\alpha) = \underline{f}'_0(\alpha) + K_{00}\overline{u}_0(\alpha)$$
$$\overline{u}'_i(\alpha) = \underline{f}_i'^{(\alpha)} + K_{ii}\underline{u}_i(\alpha) + \frac{h}{2}J_{i0}\overline{u}_0(\alpha) + h\sum_{j=1}^{k} J_{ij}\,\overline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} J_{ij}\,\underline{u}_j(\alpha) + \frac{h}{2}J_{ii}\underline{u}_i(\alpha) \qquad (41)$$

From systems (18) and (32) the following system was obtained;

$$\left.\begin{aligned}
-\underline{f}_0(\alpha) &= -\underline{u}_0(\alpha)\\
-\underline{f}_i(\alpha) &= \frac{h}{2}K_{i0}\,\underline{u}_0(\alpha) + h\sum_{j=1}^{k} K_{ij}\,\underline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} K_{ij}\,\overline{u}_j(\alpha) + \frac{h}{2}K_{ii}\overline{u}_i(\alpha)\\
&\quad + \frac{h^2}{12}\left[L_{i0}\,\underline{u}_0(\alpha) + K_{i0}\,\underline{u}_0'(\alpha) - L_{ii}\,\overline{u}_i(\alpha) + K_{ii}\overline{u}_i'(\alpha)\right] - \underline{u}_i(\alpha)\\
-\underline{f}'_0(\alpha) &= K_{00}\underline{u}_0(\alpha) - \underline{u}'_0(\alpha)\\
-\underline{f}_i'(\alpha) &= K_{ii}\overline{u}_i(\alpha) + \frac{h}{2}J_{i0}\underline{u}_0(\alpha) + h\sum_{j=1}^{k} J_{ij}\,\underline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} J_{ij}\,\overline{u}_j(\alpha) + \frac{h}{2}J_{ii}\overline{u}_i(\alpha) - \underline{u}'_i(\alpha)
\end{aligned}\right\} \quad (42)$$

and from systems (19) and (33) the following system was obtained ;

$$\left.\begin{aligned}
-\overline{f}_0(\alpha) &= -\overline{u}_0(\alpha)\\
-\overline{f}_i(\alpha) &= \frac{h}{2}K_{i0}\,\overline{u}_0(\alpha) + h\sum_{j=1}^{k} K_{ij}\,\overline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} K_{ij}\,\underline{u}_j(\alpha) + \frac{h}{2}K_{ii}\,\underline{u}_i(\alpha)\\
&\quad + \frac{h^2}{12}\left[L_{i0}\,\overline{u}_0(\alpha) + K_{i0}\,\overline{u}_0'(\alpha) - L_{ii}\,\underline{u}_i(\alpha) + K_{ii}\,\underline{u}_i'(\alpha)\right] - \overline{u}_i(\alpha)\\
-\overline{f}'_0(\alpha) &= -\overline{u}'_0(\alpha) + K_{00}\overline{u}_0(\alpha)\\
-\overline{f}'_i(\alpha) &= K_{ii}\underline{u}_i(\alpha) + \frac{h}{2}J_{i0}\overline{u}_0(\alpha) + h\sum_{j=1}^{k} J_{ij}\,\overline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} J_{ij}\,\underline{u}_j(\alpha) + \frac{h}{2}J_{ii}\underline{u}_i(\alpha) - \overline{u}'_i(\alpha)
\end{aligned}\right\} \quad (43)$$

The systems (42) and (43) are of $2(n+1)$ equations with $2(n+1)$ unknowns. Collecting these two systems together leads the following system of $4(n+1)$ equations with $4(n+1)$ unknowns:

$$\left.\begin{aligned}
-\underline{f}_0(\alpha) &= -\underline{u}_0(\alpha)\\
-\underline{f}_i(\alpha) &= \frac{h}{2}K_{i0}\,\underline{u}_0(\alpha) + h\sum_{j=1}^{k} K_{ij}\,\underline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} K_{ij}\,\overline{u}_j(\alpha) + \frac{h}{2}K_{ii}\overline{u}_i(\alpha)\\
&\quad + \frac{h^2}{12}\left[L_{i0}\,\underline{u}_0(\alpha) + K_{i0}\,\underline{u}_0'(\alpha) - L_{ii}\,\overline{u}_i(\alpha) + K_{ii}\overline{u}_i'(\alpha)\right] - \underline{u}_i(\alpha)\\
-\underline{f}'_0(\alpha) &= K_{00}\underline{u}_0(\alpha) - \underline{u}'_0(\alpha)\\
-\underline{f}_i'(\alpha) &= K_{ii}\overline{u}_i(\alpha) + \frac{h}{2}J_{i0}\underline{u}_0(\alpha) + h\sum_{j=1}^{k} J_{ij}\,\underline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} J_{ij}\overline{u}_j(\alpha) + \frac{h}{2}J_{ii}\overline{u}_i(\alpha) - \underline{u}'_i(\alpha)\\
-\overline{f}_0(\alpha) &= -\overline{u}_0(\alpha)\\
-\overline{f}_i(\alpha) &= \frac{h}{2}K_{i0}\,\overline{u}_0(\alpha) + h\sum_{j=1}^{k} K_{ij}\,\overline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} K_{ij}\,\underline{u}_j(\alpha) + \frac{h}{2}K_{ii}\,\underline{u}_i(\alpha)\\
&\quad + \frac{h^2}{12}\left[L_{i0}\,\overline{u}_0(\alpha) + K_{i0}\,\overline{u}_0'(\alpha) - L_{ii}\,\underline{u}_i(\alpha) + K_{ii}\,\underline{u}_i'(\alpha)\right] - \overline{u}_i(\alpha)\\
-\overline{f}'_0(\alpha) &= -\overline{u}'_0(\alpha) + K_{00}\overline{u}_0(\alpha)\\
-\overline{f}'_i(\alpha) &= K_{ii}\underline{u}_i(\alpha) + \frac{h}{2}J_{i0}\overline{u}_0(\alpha) + h\sum_{j=1}^{k} J_{ij}\,\overline{u}_j(\alpha) + h\sum_{j=k+1}^{i-1} J_{ij}\,\underline{u}_j(\alpha) + \frac{h}{2}J_{ii}\underline{u}_i(\alpha) - \overline{u}'_i(\alpha)
\end{aligned}\right\} \quad (44)$$

EFFICIENCY OF THE METHOD

In this section, two numerical examples were selected to reveal the efficiency of this technique. Let D_i , $0 \leq i \leq n$ denotes the error between exact and approximate solutions at x_i obtained by using the distance defined by relation (3).

Example 4.1

Consider the linear Volterra fuzzy integral equation (13) with

$$\underline{f}(x,\alpha) = \alpha x - x^2[\tfrac{2}{3}\alpha x^3 - \tfrac{4}{3}x^3 - \tfrac{1}{2}\alpha x^2 + x^2 + \tfrac{1}{12}\alpha - \tfrac{1}{12}]$$

$$\overline{f}(x,\alpha) = (2-\alpha)x - x^2[\tfrac{2}{3}\alpha x^3 - \tfrac{1}{2}\alpha x^2 + \tfrac{1}{12}\alpha - \tfrac{1}{12}]$$

and the kernel

$$K(x,t) = x^2(1-2t), \quad 0 \le t \le x, \ 0 \le x \le 1$$

where $n = 3$, $h = \frac{1}{3}$, $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$.

The exact solution in this case is given by

$$\underline{u}(x,\alpha) = \alpha x$$

$$\overline{u}(x,\alpha) = (2-\alpha)x$$

From system (44) the following 16×16 algebraic system was obtained;

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} \underline{U} \\ \overline{U} \end{bmatrix} = \begin{bmatrix} \underline{F} \\ \overline{F} \end{bmatrix}$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0185 & -0.9856 & 0 & 0 & 0.001 & 0.0003 & 0 & 0 \\ 0.0741 & 0.0494 & -0.9918 & 0 & 0.0041 & 0 & 0 & 0 \\ 0.1667 & 0.1111 & 0 & -0.9918 & 0.0093 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0.1111 & 0.0740 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0.2222 & 0.1481 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0.3333 & 0.2222 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0021 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0082 & 0 & -0.0247 & 0 & 0 & 0 & -0.0014 & 0 \\ -0.0185 & 0 & -0.1111 & -0.1667 & 0 & 0 & 0 & 0.0093 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2222 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2222 & -1.3333 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\underline{U} = [\underline{u}(x_0,\alpha), \underline{u}(x_1,\alpha), \underline{u}(x_2,\alpha), \underline{u}(x_3,\alpha), \underline{\acute{u}}(x_0,\alpha), \underline{\acute{u}}(x_1,\alpha), \underline{\acute{u}}(x_2,\alpha), \underline{\acute{u}}(x_3,\alpha)]^T,$$

$$\overline{U} = [\overline{u}(x_0,\alpha), \overline{u}(x_1,\alpha), \overline{u}(x_2,\alpha), \overline{u}(x_3,\alpha), \overline{\acute{u}}(x_0,\alpha), \overline{\acute{u}}(x_1,\alpha), \overline{\acute{u}}(x_2,\alpha), \overline{\acute{u}}(x_3,\alpha)]^T,$$

$$\underline{F} = \left[-\underline{f}(x_0,\alpha), -\underline{f}(x_1,\alpha), -\underline{f}(x_2,\alpha), -\underline{f}(x_3,\alpha), -\underline{\acute{f}}(x_0,\alpha), -\underline{\acute{f}}(x_1,\alpha), -\underline{\acute{f}}(x_2,\alpha), -\underline{\acute{f}}(x_3,\alpha)\right]^T,$$

$$\overline{F} = \left[-\overline{f}(x_0,\alpha), -\overline{f}(x_1,\alpha), -\overline{f}(x_2,\alpha), -\overline{f}(x_3,\alpha), -\overline{\acute{f}}(x_0,\alpha), -\overline{\acute{f}}(x_1,\alpha), -\overline{\acute{f}}(x_2,\alpha), -\overline{\acute{f}}(x_3,\alpha)\right]^T.$$

When solving the above system, the solution for different values of α was obtained and the errors $D_i, i = 0, \frac{1}{3}, \frac{2}{3}, 1$ as shown in table (1).

Table 1: The Numerical Results and Error of Example (4.1)

U-Values	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
$\underline{u}(x_0, \alpha)$	0	0	0	0	0	0	0	0	0	0	0
$\underline{u}(x_1, \alpha)$	0.0024	0.0358	0.0692	0.1025	0.1359	0.1693	0.2026	0.2360	0.2694	0.3027	0.3361
$\underline{u}(x_2, \alpha)$	-0.0211	0.0474	0.1160	0.1845	0.2530	0.3215	0.3900	0.4585	0.5271	0.5956	0.6641
$\underline{u}(x_3, \alpha)$	-0.0537	0.0508	0.1553	0.2598	0.3643	0.4688	0.5732	0.6777	0.7822	0.8867	0.9912
$\underline{\acute{u}}(x_0, \alpha)$	0	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000	1.0000
$\underline{\acute{u}}(x_1, \alpha)$	-0.0101	0.0901	0.1903	0.2905	0.3907	0.4909	0.5911	0.6913	0.7915	0.8917	0.9919
$\underline{\acute{u}}(x_2, \alpha)$	-0.0566	0.0458	0.1483	0.2508	0.3532	0.4557	0.5582	0.6606	0.7631	0.8656	0.9681
$\underline{\acute{u}}(x_3, \alpha)$	-0.1805	-0.0686	0.0434	0.1553	0.2673	0.3792	0.4912	0.6031	0.7151	0.8270	0.9389
$\overline{u}(x_0, \alpha)$	0	0	0	0	0	0	0	0	0	0	0
$\overline{u}(x_1, \alpha)$	0.6697	0.6364	0.6030	0.5696	0.5363	0.5029	0.4695	0.4362	0.4028	0.3694	0.3361
$\overline{u}(x_2, \alpha)$	1.3492	1.2807	1.2122	1.1437	1.0752	1.0067	0.9381	0.8696	0.8011	0.7326	0.6641
$\overline{u}(x_3, \alpha)$	2.0360	1.9315	1.8270	1.7225	1.6181	1.5136	1.4091	1.3046	1.2001	1.0956	0.9912
$\overline{\acute{u}}(x_0, \alpha)$	2.0000	1.9000	1.8000	1.7000	1.6000	1.5000	1.4000	1.3000	1.2000	1.1000	1.0000
$\overline{\acute{u}}(x_1, \alpha)$	1.9940	1.8938	1.7936	1.6934	1.5932	1.4930	1.3928	1.2926	1.1924	1.0922	0.9919
$\overline{\acute{u}}(x_2, \alpha)$	1.9928	1.8903	1.7878	1.6853	1.5829	1.4804	1.3779	1.2755	1.1730	1.0705	0.9681
$\overline{\acute{u}}(x_3, \alpha)$	2.0584	1.9464	1.8345	1.7225	1.6106	1.4987	1.3867	1.2748	1.1628	1.0509	0.9389
x_i	0	$\frac{1}{3}$	$\frac{2}{3}$	1							
D_i	0	0.0028	0.0026	0.0088							

Example 4.2: Consider the linear Volterra fuzzy integral equation (13) with

$$\underline{f}(x,\alpha) = \frac{1}{30} sin(x/2)\,[16 + 22\alpha + 26\alpha^2 - 4\alpha^3] + \frac{1}{30} sin^2(x/2)[-12 + 3\alpha + 3\alpha^3] + \frac{1}{30} sin(x/2)\, sin(3x/2)\,[4 - \alpha - \alpha^3]$$

$$\overline{f}(x,\alpha) = \frac{1}{30} sin(x/2)[104 + 22\alpha + 4\alpha^2 - 26\alpha^3] + \frac{1}{30} sin^2(x/2)[-3\alpha - 3\alpha^2] + \frac{1}{30} sin(x/2)\, sin(3x/2)\,[\alpha + \alpha^2]$$

and the kernel

$$K(x,t) = 0.1\, sin(t) \sin(x/2), \quad 0 \le t \le x, \quad 0 \le x \le 2\pi$$

where $n = 3$, $h = \frac{2\pi}{3}$, $x_0 = 0$, $x_1 = \frac{2\pi}{3}$, $x_2 = \frac{4\pi}{3}$, $x_3 = 2\pi$.

The exact solution in this case is given by:

$$\underline{u}(x,\alpha) = (\alpha + \alpha^2) \sin(x/2)$$

$$\overline{u}(x,\alpha) = (4 - \alpha - \alpha^3) \sin(x/2)$$

From system (44) the following 16×16 algebraic system was obtained;

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} \underline{U} \\ \overline{U} \end{bmatrix} = \begin{bmatrix} \underline{F} \\ \overline{F} \end{bmatrix}$$

Where;

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0317 & -0.9057 & 0 & 0 & 0 & 0.0274 & 0 & 0 \\ 0.0317 & 0.1571 & -0.9842 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0.0750 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.0785 & 0 & 0 & 0 & -0.0274 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0227 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0453 & -0.0523 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0907 & 0.0907 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\underline{U}$, $\overline{U}$, $\underline{F}$, $\overline{F}$ are described in the previous example.

The solution of the above system for various values of α and the errors D_i, $i = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ are shown in table (2).

Table 2: The Numerical Results of Example (4.2) with its Errors

U-Values	$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
$\underline{u}(x_0, \alpha)$	0	0	0	0	0	0
$\underline{u}(x_1, \alpha)$	0.1750	0.3668	0.6235	0.9430	1.3231	1.7619
$\underline{u}(x_2, \alpha)$	-0.0700	0.1493	0.4413	0.8068	1.2468	1.7620
$\underline{u}(x_3, \alpha)$	0	0	0	0	0	0
$\underline{\acute{u}}(x_0, \alpha)$	0.2667	0.3568	0.4784	0.6283	0.8032	1.0000
$\underline{\acute{u}}(x_1, \alpha)$	-0.1239	-0.0502	0.0475	0.1706	0.3206	0.4989
$\underline{\acute{u}}(x_2, \alpha)$	-0.1266	-0.1723	-0.2338	-0.3098	-0.3986	-0.4988
$\underline{\acute{u}}(x_3, \alpha)$	-0.2444	-0.3370	-0.4618	-0.6159	-0.7962	-1.0000
$\overline{u}(x_0, \alpha)$	0	0	0	0	0	0
$\overline{u}(x_1, \alpha)$	3.3487	3.1851	2.9848	2.7076	2.3134	1.7619
$\overline{u}(x_2, \alpha)$	3.5941	3.4030	3.1673	2.8441	2.3900	1.7620
$\overline{u}(x_3, \alpha)$	0	0	0	0	0	0
$\overline{\acute{u}}(x_0, \alpha)$	1.7333	1.6592	1.5696	1.4437	1.2608	1.0000
$\overline{\acute{u}}(x_1, \alpha)$	1.1218	1.0560	0.9743	0.8632	0.7092	0.4989
$\overline{\acute{u}}(x_2, \alpha)$	-0.8709	-0.8332	-0.7876	-0.7236	-0.6308	-0.4988
$\overline{\acute{u}}(x_3, \alpha)$	-1.7556	-1.6789	-1.5861	-1.4561	-1.2677	-1.0000
x_i	0	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	2π		
D_i	0	0.0298	0.0300	2.4493×10^{-16}		

CONCLUSIONS

In this paper, the modified trapezoidal method was presented to solve linear Volterra fuzzy integral equations of the second kind. The efficiency and simplicity of this method are illustrated by introducing two numerical examples with known exact solutions. All calculations in this paper have been achieved using MATLAB software.

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